WHEN THE BULLWHIP EFFECT IS AN INCREASING FUNCTION OF THE LEAD TIME

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Gerard Gaalman[1] Stephen M. Disney[2] Xun Wang[2]

[1]Department of Operations, Faculty of Economics and Business, University of Groningen, The Netherlands

[2]Logistics Systems Dynamics Group, Cardiff Business School, Cardiff University.

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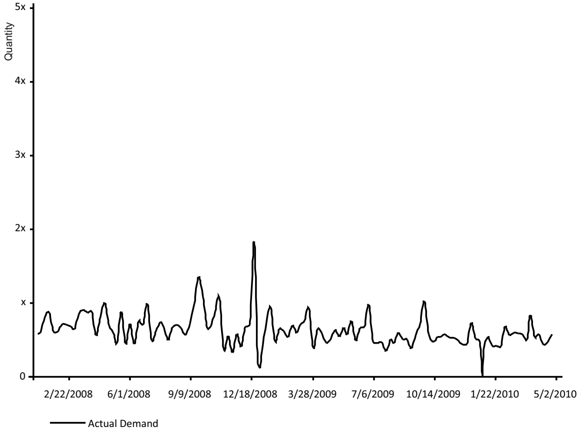
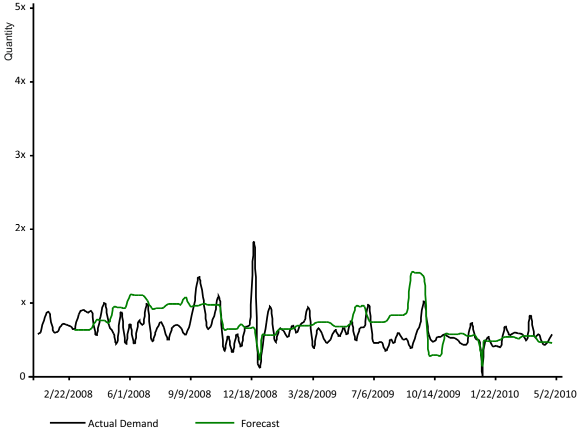
Supply chain demand can be variable

Here is a real life demand pattern.



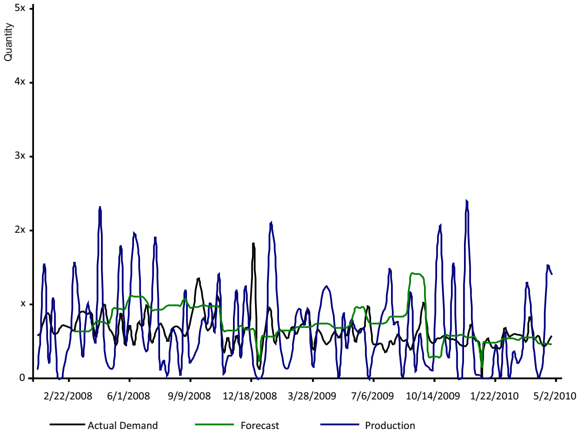
Company forecasts can be biased

This is how the company forecasted the demand pattern one week ahead.

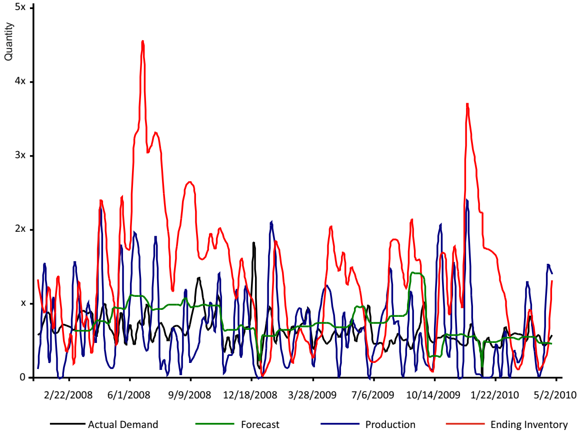


Forecasting and replenishment rules create bullwhip

The production was 7 times more variable than demand.



The inventory was 16 times more variable then demand.



Does the bullwhip effect always increase in the lead-time?

- The bullwhip effect has been extensively studied since Lee et al. (1997).
- Dejonckheere et al. (2003) considers the link between lead times and the bullwhip effect. They showed that for all demand processes, for all lead times, the OUT replenishment policy, with exponential smoothing and moving average forecasts, always generates bullwhip.
- **However, in general little is known about the interaction between the bullwhip effect and the lead-time.**
- Our contribution is to determine the conditions under which the bullwhip effect increases in the lead time under ARMA(p,q) demand with MMSE forecasts.
- We also determine for ARMA(2,2) demand (a class of second-order discrete time systems), when the system has a non-negative impulse response.

Our increasing in the lead time bullwhip story

- z-transform of the ARMA(2,2) demand process
- The order-up-to replenishment policy
- The bullwhip criterion
- Tsypkin's relation for calculating variances from impulse responses
- The demand and order variances
- Necessary and sufficient condition that increasing bullwhip in the lead time requires a positive demand impulse response
- Complete characterisation of the positivity of the impulse response for the six possible eigenvalue orders for ARMA(2,2) demand and by this, all second order control systems

The ARMA(2,2) demand process

- Ali et al. (2012) found that 75% of 1798 different SKU's in a European retailer belonged to, or were sub-sets of, the ARMA(2,2) demand process, Box et al. (2008).
- The ARMA(2,2) process is given by

$$d_t = \mu_d + \sum_{i=1}^{2} \phi_i \left(d_{t-i} - \mu_d\right) - \sum_{j=1}^{2} \theta_j \epsilon_{t-j} + \epsilon_t. \tag{1}$$

- Here, d_t is the demand in time period t, μ_d is the mean demand, ϕ_i are the auto-regressive coefficients, θ_j are the moving average coefficients, and ϵ_t is a stochastic independent and identically distributed (i.i.d.) random variable with zero mean and variance σ_ϵ^2.
- The z-transform transfer function of the ARMA(2,2) demand process is given by

$$D[z] = \frac{B[z]}{A[z]} = \frac{z^2 - \theta_1 z - \theta_2}{z^2 - \phi_1 z - \phi_2}. \tag{2}$$

The order-up-to replenishment policy

- The order-up-to (OUT) policy is a popular policy for placing production and replenishment orders to maintain control over an inventory.
- The OUT policy is available native in many commercial ERP/MRP systems; often used to to schedule high volume, long life products.
- The order-up-to policy, creates replenishment orders, o_t, via

$$o_t = \hat{d}_{t+k+1|t} - (i_t - \mu_i) - \sum_{j=1}^{k} \left(o_{t-j} - \hat{d}_{t+j|t} \right), \tag{3}$$

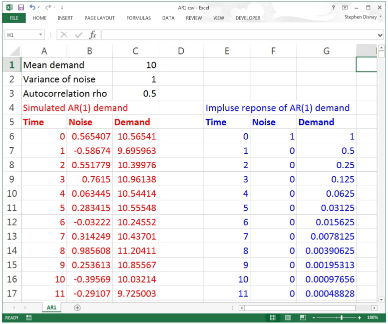
- $\hat{d}_{t+k+1|t}$ is a minimum mean squared error (MMSE) forecast of the demand in period $t+k+1$ conditional upon the information available at time t, Box et al. (2008).
- μ_i is the mean inventory, or safety stock.
- The inventory balance equation completes the definition of the OUT policy,

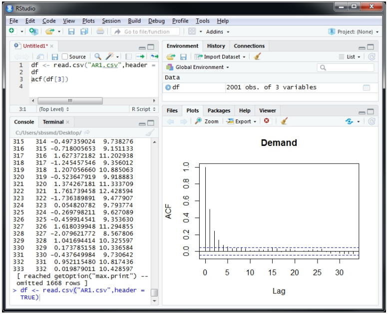
$$i_{t+1} = i_t + o_{t-k} - d_{t+1}. \tag{4}$$

- $k \in \mathbb{N}^+$ is the replenishment lead time.

The autocorrelation function and the impulse response

As we have a linear system, the superposition principle implies the system's impulse response is the same as the system's autocorrelation function.





The bullwhip criterion

- The usual way to measure bullwhip effect is the ratio, BI,

$$BI = (\sigma_o^2/\sigma_d^2) > 1 \tag{5}$$

where σ_o^2 is the variance of the replenishment orders o_t and σ_d^2 is the variance of the demand, d_t.

- These variances only exist is when demand is stationary. When demand becomes non-stationary, (5) suggests that $BI = 1$ and bullwhip is not present, but this is not true when demand is non-stationary, or near non-stationary.
- The following bullwhip criterion $CB[k]$ provides a better measure,

$$CB[k] = (\sigma_o^2 - \sigma_d^2)/\sigma_\epsilon^2. \tag{6}$$

- When $CB[k] > 0$, a bullwhip effect exists; when $CB[k] < 0$ the orders have less variance than the demand.

Variances and the sum of the squared impulse response

- Both the variance of the demand and the variance of the orders are required to determine whether bullwhip exists.
- How might we obtain these?
- The impulse response function directly allows the exact computation of the variance of the system output:

Lemma 1. Tsypkin's Relation

If the input x_t to a linear system with impulse response function g_t is an i.i.d. random process with variance σ_x^2, then the long-run variance of the output y_t is

$$\sigma_y^2 = \sigma_x^2 \sum_{t=0}^{\infty} (\tilde{g}_t)^2, \tag{7}$$

(Tsypkin, 1964).

The ARMA(2,2) demand impulse response

- A rational transfer function can be represented in zero-pole form,

$$D[z] = \frac{\prod_{i=1}^{2}(z - \lambda_i^{\theta})}{\prod_{i=1}^{2}(z - \lambda_i^{\phi})} \tag{8}$$

where $\{\lambda_i^{\theta}, \lambda_i^{\phi}\}$ are the zeros and poles (eigenvalues) of the transfer function.

- The eigenvalues of the ARMA(2,2) demand process are

$$\left\{\lambda_1^{\theta} = \frac{1}{2}\left(\theta_1 - \sqrt{\theta_1^2 + 4\theta_2}\right), \lambda_2^{\theta} = \frac{1}{2}\left(\theta_1 + \sqrt{\theta_1^2 + 4\theta_2}\right)\right\} \tag{9}$$

and

$$\left\{\lambda_1^{\phi} = \frac{1}{2}\left(\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}\right), \lambda_2^{\phi} = \frac{1}{2}\left(\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}\right)\right\}, \tag{10}$$

Gaalman et al. (2018).

- Note, the poles and zeros can be real, (conjugate) complex, and can have common poles or zeros.

Impulse response of the ARMA(2,2) demand

Lemma 2: Impulse response of the ARMA(2,2) demand

The ARMA(2,2) demand impulse response is

$$\tilde{d}_t = \begin{cases} 1, \text{ if } t = 0, \\ r_1(\lambda_1^\phi)^{t-1} + r_2(\lambda_2^\phi)^{t-1}, \text{ if } t \geq 1, \end{cases} \tag{11}$$

where,

$$r_1 = \frac{(\lambda_1^\phi - \lambda_1^\theta)(\lambda_1^\phi - \lambda_2^\theta)}{(\lambda_1^\phi - \lambda_2^\phi)} \text{ and } r_2 = \frac{(\lambda_2^\phi - \lambda_1^\theta)(\lambda_2^\phi - \lambda_2^\theta)}{(\lambda_2^\phi - \lambda_1^\phi)}. \tag{12}$$

Impulse response of the orders

Lemma 3: Impulse response of the orders

The impulse response of the orders is given by

$$\tilde{o}_t = \begin{cases} \sum_{j=0}^{k+1} \tilde{d}_{t+j}, & \text{if } t = 0, \\ \tilde{d}_{t+k+1}, & \text{if } t > 0. \end{cases} \tag{13}$$

Proof Under the order-up-to policy,

$$o_t = d_t + \sum_{j=1}^{k+1} \hat{d}_{t+j|t} - \sum_{j=1}^{k+1} \hat{d}_{t+j|t-1},$$

When demand as an ARMA(2,2) impulse response, $d_0 = \tilde{d}_0$ and $\hat{d}_{t+j|t} = \tilde{d}_{t+j}$ for $t \geqslant 0$, otherwise $\hat{d}_{t+j|t} = 0$. The consequences of these facts lead to the stated relations in (13).

The demand and order variances

- Using Tsypkin's relation, the demand variance is

$$\sigma_d^2 = \sigma_\epsilon^2 \sum_{t=0}^{\infty} \tilde{d}_t^2. \tag{14}$$

- The order variance is

$$\sigma_o^2 = \sigma_\epsilon^2 \left(\left(\sum_{j=0}^{k+1} \tilde{d}_j \right)^2 + \sum_{t=1}^{\infty} \tilde{d}_{t+k+1}^2 \right). \tag{15}$$

- Using these variances, $CB[k]$ becomes

$$CB[k] = \frac{\sigma_o^2 - \sigma_d^2}{\sigma_\epsilon^2} = \left(\sum_{j=0}^{k+1} \tilde{d}_j \right)^2 - \sum_{t=0}^{k+1} \tilde{d}_t^2. \tag{16}$$

A necessary and sufficient condition for an increasing in the lead time bullwhip effect

Theorem 4.

Iff $\{\tilde{d}_1, \tilde{d}_2, ..., \tilde{d}_{k+1}\} > 0$ then $CB[k]$ is positive and increasing in the lead time.

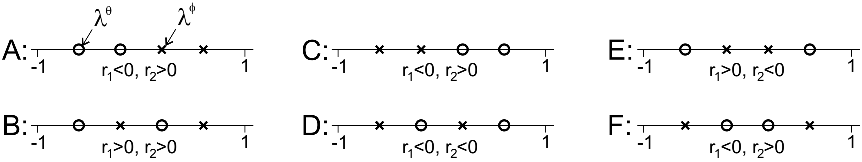
Proof $CB[k]$ is positive and increasing in k if $CB[0] > 0$ and $\forall k$, $CB[k] - CB[k-1] > 0$

- Note always, $\tilde{d}_0 = 1$.
- $CB[0] = \left(\sum_{j=0}^{1} \tilde{d}_j\right)^2 - \sum_{t=0}^{1} \tilde{d}_t^2 = 2\tilde{d}_0\tilde{d}_1$ is positive if additionally $\tilde{d}_1 > 0$
- $CB[1] - CB[0] = 2(\tilde{d}_0 + \tilde{d}_1)\tilde{d}_2$ is positive if additionally $\tilde{d}_2 > 0$
- $CB[2] - CB[1] = 2(\tilde{d}_0 + \tilde{d}_1 + \tilde{d}_2)\tilde{d}_3$ is positive if additionally $\tilde{d}_3 > 0$
- This process can be continued for all k. $\square$

Bullwhip is always increasing in the lead-time iff the demand impulse response is positive for all t.

Note. This result holds for all ARMA(p,q) demand processes

ARMA(2,2) demand has six possible eigenvalue orderings



Remember Lemma 2?

The ARMA(2,2) demand impulse response is

$$\tilde{d}_t = \begin{cases} 1, \text{ if } t = 0, \\ r_1(\lambda_1^{\phi})^{t-1} + r_2(\lambda_2^{\phi})^{t-1}, \text{ if } t \geq 1, \end{cases} \tag{11}$$

where,

$$r_1 = \frac{(\lambda_1^{\phi} - \lambda_1^{\theta})(\lambda_1^{\phi} - \lambda_2^{\theta})}{(\lambda_1^{\phi} - \lambda_2^{\phi})} \text{ and } r_2 = \frac{(\lambda_2^{\phi} - \lambda_1^{\theta})(\lambda_2^{\phi} - \lambda_2^{\theta})}{(\lambda_2^{\phi} - \lambda_1^{\phi})}. \tag{12}$$

Case A: Eigenvalue order is $-1 < \mathrm{Re}[\lambda_1^\theta] \leq \mathrm{Re}[\lambda_2^\theta] < \lambda_1^\phi \leq \lambda_2^\phi < 1$.

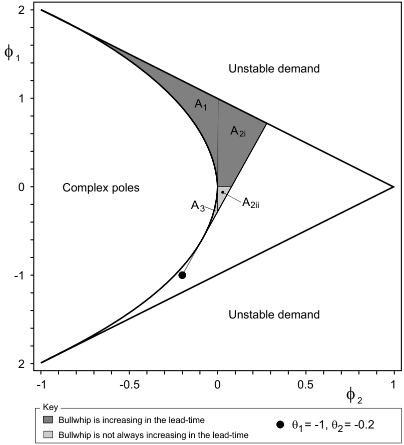
It is easy to verify that $r_1 < 0 < r_2$, $\tilde{d}_1 > 0$, and $-r_2/r_1 > 1$. This case can exist when complex zeros are present. Depending of the sign of the poles, $\{\lambda_1^\phi, \lambda_2^\phi\}$, we need to consider the following three sub-cases:

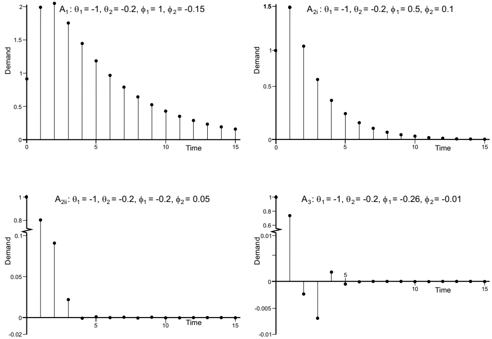
- **Case** A_1**:** $0 < \lambda_1^\phi \leq \lambda_2^\phi$. Using $r_1 = \tilde{d}_1 - r_2$ in (11) provides

$$\tilde{d}_{t+1} = \tilde{d}_1(\lambda_1^\phi)^t + r_2((\lambda_2^\phi)^t - (\lambda_1^\phi)^t) > 0, \tag{17}$$

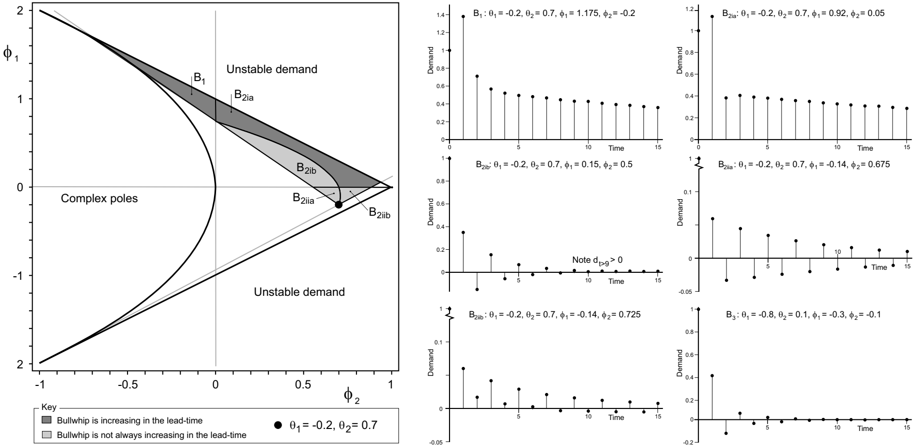
  which is positive for all t as $\tilde{d}_1, r_2, \lambda_1^\phi, \lambda_2^\phi > 0$ and $\lambda_2^\phi > \lambda_1^\phi$. This means that the bullwhip effect is increasing in the lead time.

Case A: Parameter hyper-plane and numerical cross-check



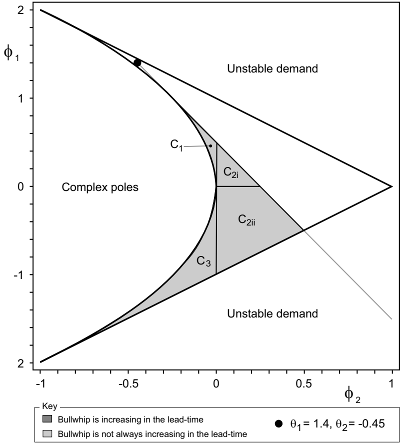


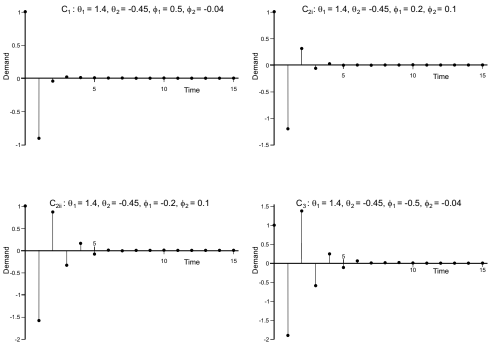
Case B: $-1 < \lambda_1^\theta < \lambda_1^\phi < \lambda_2^\theta < \lambda_2^\phi < 1$



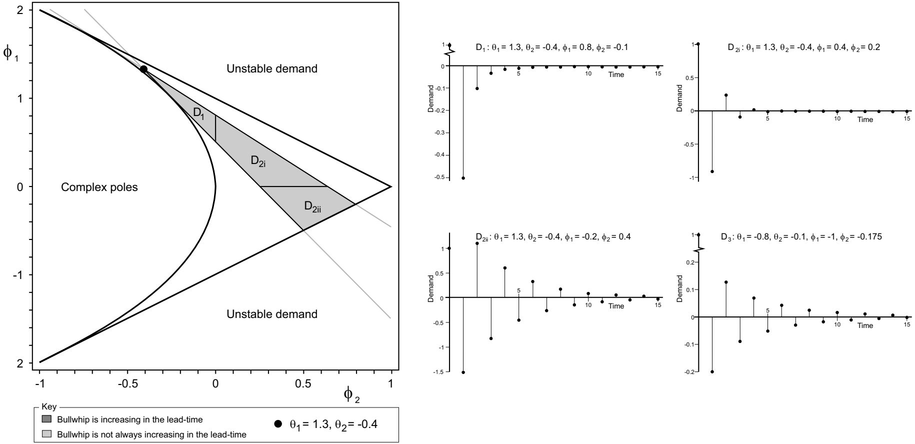
Note: It is not possible to illustrate all possible subsets of our 4-D parameter space on a 2-D map. Hence case B_3 is not shown on the parameter hyper-plane.

Case C: $\lambda_1^\phi \leq \lambda_2^\phi < \mathrm{Re}[\lambda_1^\theta] \leq \mathrm{Re}[\lambda_2^\theta]$



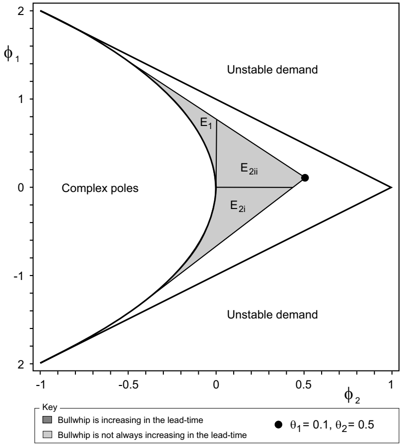


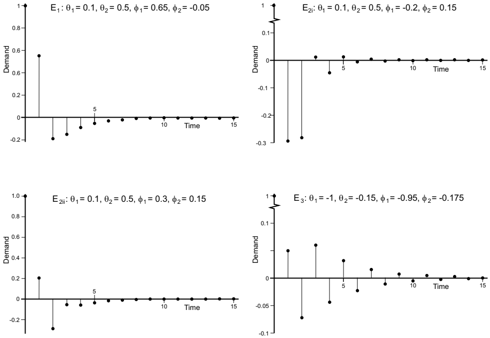
Case D: $\lambda_1^{\phi} < \lambda_1^{\theta} < \lambda_2^{\phi} < \lambda_2^{\theta}$



Note: It is not possible to illustrate all possible subsets of our 4-D parameter space on a 2-D map. Hence case d_3 is not shown on the parameter hyper-plane.

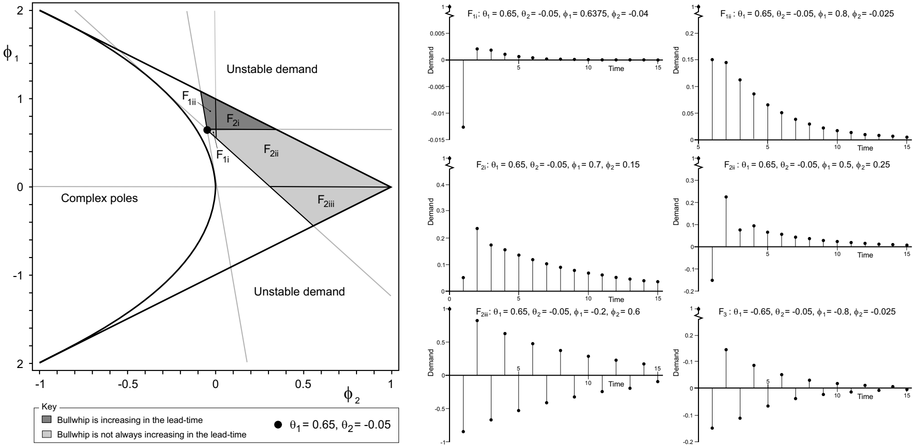
Case E: $\lambda_1^\theta < \lambda_1^\phi \le \lambda_2^\phi < \lambda_2^\theta$





Note: It is not possible to illustrate all possible subsets of our 4-D parameter space on a 2-D map. Hence case E_3 is not shown on the parameter hyper-plane.

Case F: $\lambda_1^{\phi} < \mathrm{Re}[\lambda_1^{\theta}] \leq \mathrm{Re}[\lambda_2^{\theta}] < \lambda_2^{\phi}$



Note: It is not possible to illustrate all possible subsets of our 4-D parameter space on a 2-D map. Hence case F_3 is not shown on the parameter hyper-plane.

Concluding remarks

- We have introduced a new bullwhip metric, $CB[k]$, useful when large order and demand variances are present.
- Theorem 4 showed the positivity of the demand impulse response determines whether bullwhip increases over the lead-time.
- We confirmed this by studying the eigenvalues, $\{\lambda_i^{\phi}, \lambda_j^{\theta}\}$, of the demand process rather than AR and MA parameters, $\{\phi_i, \theta_j\}$, directly. This was efficient as only the order of the eigenvalues determines a lead-time/bullwhip relationship, not the specific value of the eigenvalues or the demand parameters.
- We illustrated our results by studying all the possible eigenvalue orderings of the ARMA(2,2) demand process.
- The ARMA(2,2) demand process is equivalent to the general class of second order discrete time control systems. Thus we have also obtained **a complete understanding of when a positive impulse response exists for all second order control systems**.

Concluding remarks

- The practicing manager may be considering a lead-time reduction.
- Depending on the demand process present there may, or may not, be a bullwhip benefit from reducing the lead time.
- If there is a benefit, the cost of reducing the lead time may be offset against the lower capacity costs, (Hosoda and Disney, 2012).
- If bullwhip does not increase in the lead time, perhaps different (cheaper, slower, more ecologically friendly) transport modes or production technology can be used instead?
- But what should the manager do if there is a bullwhip benefit from reducing a short lead-time, but for long lead times there is no bullwhip benefit?
- Is it possible to find empirical examples where there is, and is not, a bullwhip benefit from a lead-time reduction?

Thank you for listening

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Gerard Gaalman, Stephen M. Disney, and Xun Wang

DisneySM@cardiff.ac.uk

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