THE NONLINEAR DYNAMICS OF ORDER-UP-TO INVENTORY SYSTEMS WITH LOST SALES

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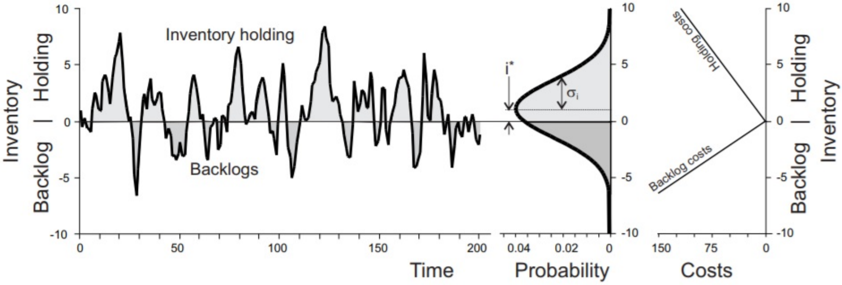
Overview

- We study the dynamic consequences of lost sales when there is insufficient inventory to satisfy demand.
- Demand is assumed to be independently and identically distributed and drawn from a normal distribution.
- We consider the industrially popular order-up-to policy with unit lead time is used to make replenishment orders.
- We obtain expressions for order and inventory distributions, the Bullwhip and Net Stock Amplification ratios, as well as the mean inventory levels and the achieved fill rate.
- We do this for when the lost sales are fully observable, and when the lost sales are unobservable.

Backlog systems are commonly assumed in the literature

The majority of inventory theory is built on *backlog systems*, where excess demand is accumulated in an order book and is delivered as soon as inventory becomes available (Zipkin, 2008; Bijvank and Vis, 2011).



... but how do customers truly act when they can't buy what they want?

How do customers actually react to a stock-out situation?

- Verbeke et al. (1998) interviewed 1,750 customers: only 20% of them would postpone the purchase.
- Gruen et al. (2002) surveyed more than 70,000 customers: only 15% of them would wait until the item is available again.
- Van Woensel et al. (2007) analysed the behaviour of 3,800 customers: only 12% of regular customers and 6% of occasional customers would delay the purchase.

Our motivation

- Many practical supply chain settings are lost-sales inventory systems.
- It seems reasonable to assume that customers' tolerance of lost sales is decreasing in an intensively competitive business context.
- However, our comprehension of the dynamic behaviour of lost-sales systems is limited compared to our understanding of backlog systems.

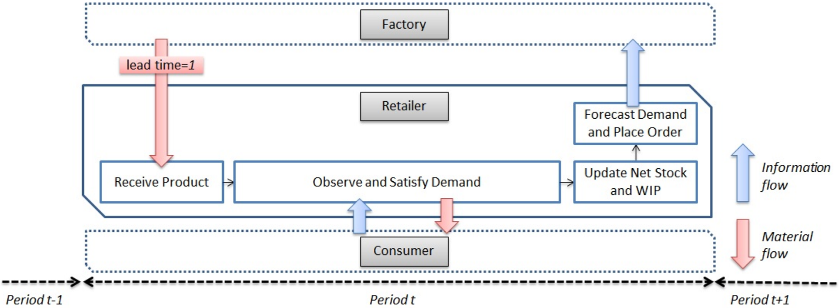
Literature on lost sales systems

- A noteworthy research stream has investigated optimal replenishment policies in lost-sales environments (Karlin and Scarf, 1958; Goldberg et al., 2016).
- It is known that complex dynamic behaviours appear in lost-sales inventory systems due to their nonlinear nature, which hampers the mathematical study (Nagatani and Helbing, 2004; Ponte et al., 2017).
- Bijvank and Vis (2011) provide a comprehensive literature review on the lost-sales inventory discipline.

Our contribution

- We investigate the impact of the lost-sales non-linearity on the dynamics of the supply chain under the well-known, industrially popular, order-up-to (OUT) replenishment policy.
- We are concerned with both the production consequences and inventory consequences of lost-sales. In particular, we consider the fundamental trade-off between order and net stock variability.
- Our results help practitioners to dynamically design lost-sales supply chains to reduce cost, improve product availability, and create a stable working environment for employees and suppliers.

Supply chain structure and sequence of events



The lost-sales inventory model (I)

Mathematical model: Flow of materials

Receipts:

$$r_t = q_{t-1} \tag{1}$$

Demand:

$$d_t = z_t : z_t \in \mathcal{N}(\mu, \sigma^2) \tag{2}$$

Inventory:

$$i_t = [i_{t-1} + r_t - d_t]^+ \tag{3}$$

Satisfied demand:

$$s_t = \min\{i_{t-1} + r_t, d_t\} \tag{4}$$

The lost-sales inventory model (II)

Mathematical model: Flow of orders

Forecast:

$$f_t = \eta \tag{5}$$

Order quantity:

$$q_t = f_t + (\delta f_t - i_t) = (1 + \delta) f_t - i_t \tag{6}$$

where:

- η (decision variable) is the demand forecast. If the retailer is able to observe the whole demand, $\eta = \mu$, resulting in a minimum mean squared error forecast (Disney et al., 2016). If, in contrast, the retailer is unable to observe the whole demand, we assume $\eta < \mu$.
- δ (decision variable) is the safety stock factor.

Performance measures (I)

Dynamic measures

Bullwhip ratio:

$$BW = \mathrm{var}(q_t)/\mathrm{var}(d_t) \tag{7}$$

Net Stock Amplification ratio:

$$NSAmp = \mathrm{var}(i_t)/\mathrm{var}(d_t) \tag{8}$$

Notes:

- In linear systems, BW is directly related to capacity costs and $NSAmp$ is directly related to inventory costs, Disney and Lambrecht (2008).
- In linear systems, the BW and $NSAmp$ metrics represent a key trade-off for managers, as it is often possible to decrease one at the cost of increasing the other, Disney et al. (2004).

Performance measures (II)

Inventory service measure

Fill rate:

$$\beta = \mathbb{E}[(s_t)^+]/\mathbb{E}[(d_t)^+] \tag{9}$$

Average inventory:

$$\tau = \mathbb{E}[i_t] \tag{10}$$

Note:

- Again, there is a key trade-off to consider here, as the fill rate can be improved at the cost of an higher average inventory.

The key step to our solution...

Theorem 1: Fundamental relations

The following *fundamental relationships* exist in our lost-sales system for the three state variables (inventory, satisfied demand, and orders):

$$i_t = [(1+\delta)\eta - d_t]^+, \tag{11}$$

$$s_t = \min\{(1+\delta)\eta, d_t\}, \tag{12}$$

$$q_t = s_t. \tag{13}$$

Proof of Theorem 1 is provided in the article.

Eqs. (11), (12) and (13) are interesting as they show the pdf's of $\{i_t, s_t, q_t\}$ can be directly obtained from the demand distribution.

Distribution of the state variables (I)

The inventory distribution

The pdf of the inventory is given by

$$\phi_i[x] = \frac{h[x]}{\sigma}\phi\left[\frac{x-(1+\delta)\eta+\mu}{\sigma}\right] + \Delta[x]\Phi\left[\frac{\mu-(1+\delta)\eta}{\sigma}\right]. \tag{14}$$

where:

- $\phi[\cdot]$ and $\Phi[\cdot]$ are respectively the pdf and cumulative distribution function (cdf) of the standard normal distribution $\mathcal{N}(0,1)$.
- $h[x] = \{1,\ \text{if } x \geq 0;\ 0,\ \text{otherwise}\}$ is the Unit Step function.
- $\Delta[x] = \{1,\ \text{if } x = 0;\ 0,\ \text{otherwise}\}$ is the Dirac Delta function.

Distribution of the state variables (II)

The distribution of the satisfied demand and the order quantity

The pdf of the satisfied demand and order quantity is given by:

$$\phi_q[x] = \phi_s[x] = \frac{h[(1+\delta)\eta - x]}{\sigma}\phi\left[\frac{x-\mu}{\sigma}\right] + \Delta[(1+\delta)\eta - x]\Phi\left[\frac{\mu-(1+\delta)\eta}{\sigma}\right] \tag{15}$$

Note, the pdf of the orders (and the pdf of the stisfied demand) is a translated reflection of the pdf of the inventory. Therefore,

$$BW = NSAmp, \tag{16}$$

which implies that the order-net stock variability trade-off is *broken* in the nonlinear, lost-sales inventory systems.

A convenient intermediate step

The expressions of the four performance indicators can be derived from the pdf's of the three state variables. Exposition of these metrics can be greatly simplified by defining the *relative safety margin*,

$$\lambda = \frac{(1+\delta)\eta - \mu}{\sigma}. \tag{17}$$

Physically λ can be interpreted the protection of the on-hand inventory against shortages, provided by the demand forecast and the safety stock factor, in relative terms to the standard deviation of the demand.

In addition, let $\gamma = \sigma/\mu$ be customer demand's coefficient of variation.

The metrics in the lost-sales inventory system

Theorem 2: Expressions for the performance indicators

The *BW* and *NSAmp* metrics are

$$BW = NSAmp = \lambda\phi\left[\lambda\right] + (\lambda^2 + 1)\Phi\left[\lambda\right] - \left(\phi\left[\lambda\right] + \lambda\Phi\left[\lambda\right]\right)^2 . \tag{18}$$

The fill rate, β, is given by

$$\beta = h[1 + \gamma\lambda]\left(1 - \frac{\phi[\lambda] + \lambda(\Phi[\lambda] - 1)}{\phi\left[\gamma^{-1}\right] + \gamma^{-1}\Phi\left[\gamma^{-1}\right]}\right), \tag{19}$$

The average inventory, τ, is

$$\tau = \mu\gamma\left(\phi\left[\lambda\right] + \lambda\Phi\left[\lambda\right]\right). \tag{20}$$

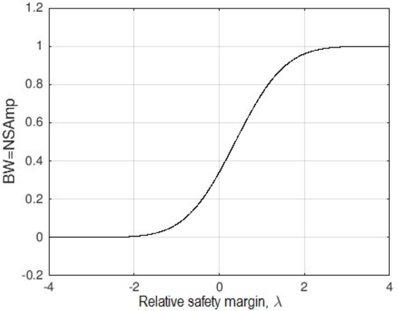
Proof of Theorem 2 is provided in the article.

Impact of the ability to observe lost sales

Following from the discussion in the mathematical model, we consider two scenarios that are relevant in practice, see Lariviere and Porteus (1999):

1. *Full demand observation* (FDO), where the retailer observes the whole customer demand, despite only satisfying a portion of it.
   - For example, this may be relevant for an internet retailer who could track customer's browsing history.
   - This is described by $\eta = \mu$, thus resulting in a simplified variant of the relative safety margin, $\lambda = \delta/\gamma$.
2. *Partial demand observation* (PDO), where excess demand becomes unobserved lost sales.
   - This occurs when customers experiencing a stock-out in store depart without leaving a trace of their disappointment; leading retailers to under-estimate the actual demand, $\eta < \mu$.
   - In such cases, employing high safety stocks would help the retailer approximate μ; an idea known as *exploration*.

Lost sales reduce bullwhip and net stock variance

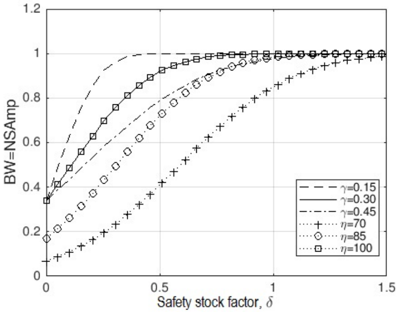


BW and *NSAmp* as a function of λ.

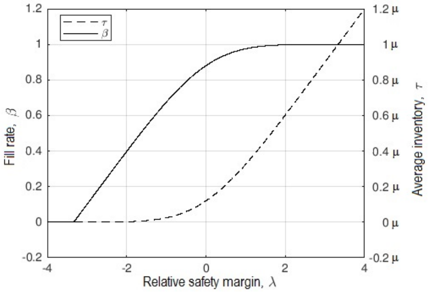
- In the backlog system, under our assumptions, $BW = 1$ and $NSAmp = 1$.
- Here, *BW* and *NSAmp* depend upon λ, a function of $\{\mu, \sigma, \delta, \eta\}$, displaying a S-shaped relationship.
- When λ is sufficiently high, the retailer rarely experiences lost sales and operates as a linear system would, while small values of λ reduce supply chain volatility.

Partial demand observations smooth the dynamics

- **FDO**. When δ is sufficiently high, $BW = 1$, $NSAmp = 1$.
- However, as δ decreases, BW and $NSAmp$ reduces.
  - Note, larger γ require greater δ to ensure linear operation.
- **PDO**. As unobserved demand grows, leading to biased forecasts, η decreases.
- BW and $NSAmp$ are very sensitive to η, both metrics decreasing as η decreases.
  - Note, η now influences BW and $NSAmp$ when $\delta = 0$.



BW and $NSAmp$ as a function of δ under FDO (without markers) and PDO (with markers, for $\gamma = 30\%$).

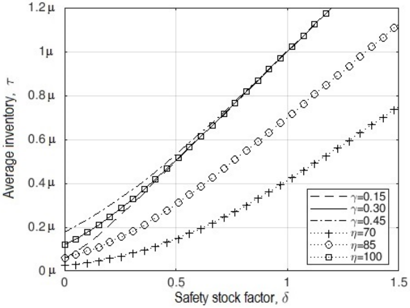


τ and β as a function of λ (for $\gamma = 30\%$).

- For $\lambda << 0$, $\tau = 0$ and $\beta = 0$.
- For larger values of λ, τ is increasing and convex in λ.
- In contrast, β is increasing and concave in λ, with $\beta = 1$ for $\lambda >> 0$.

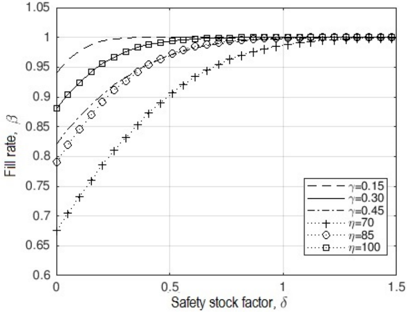
Inventory service measures insights (II)

Average inventory, τ



τ as a function of δ under FDO (without markers) and PDO (with markers, for $\gamma = 30\%$).

Fill rate, β



β as a function of δ under FDO (without markers) and PDO (with markers, for $\gamma = 30\%$).

Conclusions

- The lost-sales condition introduces behavioural differences that should be carefully considered. Ignoring lost sales in real-world environments is highly risky: using inventory configurations derived from backlog settings may result in a dramatically decreased service level.
- While the simultaneous consideration of *BW* and *NSAmp* provides a complete picture of the performance of replenishment policies in linear systems, the interpretation of *NSAmp* is different in nonlinear lost-sales settings.
- Unobserved demand in lost-sales systems helps to reduce the Bullwhip Effect. However, this dynamic improvement comes at the expense of lowering fill rates and increasing the required safety stock.

Avenues for future work

- Understanding the general lead time case.
- Considering the dynamics induced by other forecasting techniques and/or replenishment policies.
- Investigating the interactions of the lost-sales condition with other non-linearities.
- Addressing the multi-echelon effects in multiple lost-sales systems.

Thank you for listening

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